

# A REMARK ON VANISHING CYCLES WITH TWO STRATA

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**ABSTRACT.** Suppose that the critical locus  $\Sigma$  of a complex analytic function  $f$  on affine space is, itself, a space with an isolated singular point at the origin  $\mathbf{0}$ , and that the Milnor number of  $f$  restricted to normal slices of  $\Sigma - \{\mathbf{0}\}$  is constant. Then, the general theory of perverse sheaves puts severe restrictions on the cohomology of the Milnor fiber of  $f$  at  $\mathbf{0}$ , and even more surprising restrictions on the cohomology of the Milnor fiber of generic hyperplane slices.

## 1. SETTINGS

Let  $\mathcal{U}$  be an open neighborhood of the origin in  $\mathbb{C}^{n+1}$ , and  $f : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$  be a complex analytic function. Let us call  $(X, 0)$  the germ of the complex analytic hypersurface defined by this function. Suppose that the critical locus  $\Sigma$  of  $f$  is, itself, smooth outside of  $\mathbf{0}$ , and let  $s := \dim \Sigma$ .

Throughout this paper, we make the following assumptions:

- (1)  $s \geq 3$ .
- (2) The Milnor number of a transverse slice of codimension  $s$  of the hypersurface  $f^{-1}(0)$  is constant along the singular set  $\Sigma$  of  $X$  outside of  $\mathbf{0}$ , and equal to  $\mu$ .
- (3) The intersection of  $\Sigma$  with a sufficiently small sphere  $S_\varepsilon$  centered at  $\mathbf{0}$  is  $(s-2)$ -connected.

Under these hypotheses, we have:

**Theorem 1.1.** *The Milnor fiber  $F_0$  of  $f$  at  $\mathbf{0}$  can have non-zero cohomology only in degrees 0,  $n-s$ ,  $n-1$  and  $n$ .*

**Corollary 1.2.** Suppose that  $s \geq 4$  and, for a generic hyperplane  $H$ , the real link  $S_\varepsilon \cap \Sigma \cap H$  of  $\Sigma \cap H$  at  $\mathbf{0}$  is  $(s-3)$ -connected. Then, the Milnor fiber  $F_H$  of  $f|_H$  at  $\mathbf{0}$  can have non-zero cohomology only in degrees 0,  $n-s$  and  $n-1$ .

## 2. AN EXACT SEQUENCE

Let  $\mathbb{Z}_{\mathcal{U}}^\bullet$  be the constant sheaf on  $\mathcal{U}$  with stalks isomorphic to the ring of integers  $\mathbb{Z}$ . If  $\phi_f$  is the functor of vanishing cycles of  $f$ , we know (see, e.g., [2], Theorem 5.2.21) that the complex  $\phi_f[-1]\mathbb{Z}_{\mathcal{U}}^\bullet[n+1]$  is a perverse sheaf (see, e.g., [1] p. 9) on  $f^{-1}(0)$ . Let  $\mathbf{P}^\bullet$  denote the restriction of this sheaf to its support  $\Sigma$ , which is the set of critical points of  $f$  inside  $f^{-1}(0)$ .

We know that, for all  $x \in \Sigma$ , we have

$$\mathbb{H}^{-k}(\mathbb{B}(x) \cap \Sigma; \mathbf{P}^\bullet) \cong H^{-k}(\mathbf{P}^\bullet)_x \cong \tilde{H}^{n-k}(F_x; \mathbb{Z}),$$

where  $F_x$  is the Milnor fiber of  $f$  at  $x$  and  $\mathbb{B}(x)$  is a sufficiently small ball (open or closed, with non-zero radius) of  $\mathbb{C}^{n+1}$  centered at  $x$ . Let  $\mathbb{B}^*(x) = \mathbb{B}(x) - \{x\}$ .

Then, we have the exact sequence in hypercohomology:

$$\begin{aligned} \rightarrow \mathbb{H}^{-k}(\mathbb{B}(x) \cap \Sigma, \mathbb{B}^*(x) \cap \Sigma; \mathbf{P}^\bullet) &\rightarrow \mathbb{H}^{-k}(\mathbb{B}(x) \cap \Sigma; \mathbf{P}^\bullet) \\ &\rightarrow \mathbb{H}^{-k}(\mathbb{B}^*(x) \cap \Sigma; \mathbf{P}^\bullet) \rightarrow \mathbb{H}^{-k+1}(\mathbb{B}(x) \cap \Sigma, \mathbb{B}^*(x) \cap \Sigma; \mathbf{P}^\bullet) \rightarrow \end{aligned}$$

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Since  $\mathbf{P}^\bullet$  is perverse, using the cosupport condition (see e.g. [1] p. 9):

$$\mathbb{H}^{-k+1}(\mathbb{B}(x) \cap \Sigma, \mathbb{B}^*(x) \cap \Sigma; \mathbf{P}^\bullet) = 0$$

for  $-k+1 < 0$ . Therefore,

$$\tilde{H}^{n-k}(F_x; \mathbb{Z}) \cong \mathbb{H}^{-k}(\mathbb{B}(x) \cap \Sigma; \mathbf{P}^\bullet) \cong \mathbb{H}^{-k}(\mathbb{B}^*(x) \cap \Sigma; \mathbf{P}^\bullet)$$

for  $-k+1 < 0$ .

### 3. TOPOLOGICAL HYPOTHESIS

Now let us suppose that the real link  $S_\varepsilon \cap \Sigma$  of the critical locus  $\Sigma$  at  $\mathbf{0}$  is  $(s-2)$ -connected; in particular, as  $s \geq 3$ ,  $S_\varepsilon \cap \Sigma$  is simply-connected.

Our assumption on the constancy of the Milnor number of  $f$ , restricted to a normal slice to  $\Sigma$ , is equivalent to saying that  $\mathbf{P}^\bullet_{|\Sigma-\{\mathbf{0}\}}$  is locally constant, with stalk cohomology  $\mathbb{Z}^\mu$  concentrated in degree  $-s$ . As  $\mathbb{B}^*(\mathbf{0}) \cap \Sigma$  is homotopy-equivalent to  $S_\varepsilon \cap \Sigma$ , which is simply-connected, it follows that  $\mathbf{P}^\bullet_{|\mathbb{B}^*(\mathbf{0}) \cap \Sigma}$  is isomorphic to the shifted constant sheaf  $(\mathbb{Z}^\mu)^\bullet_{\mathbb{B}^*(\mathbf{0}) \cap \Sigma}[s]$ .

This implies that

$$\mathbb{H}^{-k}(\mathbb{B}^*(\mathbf{0}) \cap \Sigma; \mathbf{P}^\bullet) \cong H^{-k+s}(\mathbb{B}^*(\mathbf{0}) \cap \Sigma; \mathbb{Z}^\mu) \cong H^{-k+s}(S_\varepsilon \cap \Sigma; \mathbb{Z}^\mu).$$

Thus, as  $S_\varepsilon \cap \Sigma$  is  $(s-2)$ -connected, we have:

$$\mathbb{H}^{-s}(\mathbb{B}^*(\mathbf{0}) \cap \Sigma; \mathbf{P}^\bullet) \cong H^0(S_\varepsilon \cap \Sigma; \mathbb{Z}^\mu) \cong \mathbb{Z}^\mu,$$

and, if  $2 \leq k \leq s-1$ :

$$\mathbb{H}^{-k}(\mathbb{B}^*(\mathbf{0}) \cap \Sigma; \mathbf{P}^\bullet) \cong H^{s-k}(S_\varepsilon \cap \Sigma; \mathbb{Z}^\mu) = 0.$$

### 4. PROOFS

Combining the results from the previous two sections, we find that, if the real link of the critical locus  $\Sigma$  at  $\mathbf{0}$  is  $(s-2)$ -connected and  $s \geq 3$ , then we have for the Milnor fiber  $F$  of  $f$  at  $\mathbf{0}$ :

$$\tilde{H}^{n-s}(F; \mathbb{Z}) \cong H^0(S_\varepsilon \cap \Sigma; \mathbb{Z}^\mu) \cong \mathbb{Z}^\mu;$$

$$\tilde{H}^k(F; \mathbb{Z}) = 0, \text{ if } k \neq n-1, n.$$

This proves the theorem.

Suppose now that, in addition to our other hypotheses,  $s \geq 4$  and, for generic hyperplanes  $H$ ,  $S_\varepsilon \cap \Sigma \cap H$  is  $(s-3)$ -connected. Then,  $f|_H$  satisfies the hypotheses of the theorem, except that  $n$  is replaced by  $n-1$  and  $s$  is replaced by  $s-1$ . Thus, for the Milnor fiber  $F_H$ :

$$\tilde{H}^{n-s}(F_H; \mathbb{Z}) \cong \mathbb{Z}^\mu;$$

$$\tilde{H}^k(F_H; \mathbb{Z}) = 0, \text{ if } k \neq n-2, n-1.$$

However, by the main result of [3], the Milnor fiber  $F$  is obtained from the Milnor fiber  $F_H$  by attaching cells in dimension  $n$ . Hence,  $\tilde{H}^{n-2}(F_H; \mathbb{Z}) \cong \tilde{H}^{n-2}(F; \mathbb{Z})$ , which we know is 0. This proves the corollary.

## 5. WHEN THE CRITICAL LOCUS IS AN ICIS

Assume that the critical locus  $\Sigma$  of  $f$  is an isolated complete intersection singularity (ICIS) of dimension  $s \geq 4$ .

For an ICIS, the real link  $S_\varepsilon \cap \Sigma$  is  $(s-2)$ -connected. In addition, for a generic hyperplane  $H$ , the critical locus of  $f|_H$ , which equals  $\Sigma \cap H$ , will also be an ICIS, but now of dimension  $s-1$ . Thus,  $S_\varepsilon \cap \Sigma \cap H$  is  $((s-1)-2)$ -connected. Therefore, we are in the situation that we have considered above.

In his preprint [5] M. Shubladze asserts that if the singular locus  $\Sigma$  of  $f$  is a complete intersection with isolated singularity at  $\mathbf{0}$  of dimension  $\geq 3$  and the Milnor number for transverse sections is 1 along  $\Sigma \setminus \{\mathbf{0}\}$ , the Milnor number of  $f$  at  $\mathbf{0}$  has cohomology possibly  $\neq 0$  only in dimensions 0,  $n-s$  and  $n$ .

The results above show that, under the hypothesis of M. Shubladze, one obtains in a general way that the cohomology of the Milnor fiber of  $f$  at  $\mathbf{0}$  is possibly  $\neq 0$  in dimension 0,  $n-s$ ,  $n-1$  and  $n$ , and a similar result as the one of M. Shubladze in dimension 0,  $n-s$ ,  $n-1$  for the cohomology of the Milnor fiber of  $f$  restricted to a general hyperplane section if  $\dim \Sigma \geq 4$ .

In light of our corollary, it seems reasonable to ask: Can every function such as that studied by Shubladze can be obtained as a generic hyperplane restriction of a function satisfying the same hypotheses?

Shubladze's proof is via deformation and does not seem to easily answer the question above.

6. WHAT IF  $S_\varepsilon \cap \Sigma$  IS A HOMOLOGY SPHERE?

One might also wonder what happens if the real link of  $\Sigma$  is  $(s-1)$ -connected. This would, in fact, imply that  $S_\varepsilon \cap \Sigma$  is a homology sphere. In this case, our earlier exact sequence immediately yields that  $\tilde{H}^{n-1}(F; \mathbb{Z}) = 0$ .

A special case of  $S_\varepsilon \cap \Sigma$  being a homology sphere would occur if  $\Sigma$  were smooth. However, in this case, Proposition 1.31 of [4] implies that the Milnor number cannot change at  $\mathbf{0}$ , i.e., we have a smooth  $\mu$ -constant family, and so the non-zero cohomology of  $F$  occurs only in degrees 0 and  $n-s$ .

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